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D. A. F. OMOJOLA
School of Mathematical and Physical Sciences, University of Lagos, Nigeria
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Abstract. (i) A new formulation of the elastic neutron–alpha (n–^{4}He) scattering using the most general form of the nuclear interaction which includes the central, non-central exchange–linear spin–orbit, tensor and quadratic spin–orbit forces is given.

(ii) The resonating group method of Wheeler is employed to construct the five-body wave functions and from the Schrödinger equation a set of simultaneous integro–differential equations is derived.

(iii) The Hamada and Johnston potentials used are expressed in Gaussian form permitting analytical integrations and leading to great economy of computing time. There are repulsive cores in all states.

(iv) An extension of n–^{4}He scattering formulation to include the D-state of the alpha wave function is discussed.

(v) The resulting integro–differential equations are then solved by the method of finite differences.

Programs for computing the direct and indirect (kernels) terms and for the solution of the equations written in Algol language were prepared for the University of London Atlas computer. The equations are then solved at different incident neutron energies.

1. Introduction

The five-body problem of n–^{4}He has been formulated using the methods proposed by Omojola (1969—to be referred to as I). The elastic scattering of n–^{4}He has been studied by several authors (Hochberg *et al.* 1954, 1955, Van Der Spuy 1956 and Kanada *et al.* 1963).

The nucleon–nucleon interaction is known to include both central and non-central forces. According to Rosenfeld (1948—pp. 312–4), and Okubo and Marshak (1958), the most general form of this interaction with the invariance and symmetry requirements must be a linear combination of the following terms:

(i) The central exchange force

(ii) The linear spin–orbit force

$$S(\mathbf{r}_{ij}) = V(\mathbf{r}_{ij})\{(\mathbf{s}_i+\mathbf{s}_j)(\mathbf{r}_i-\mathbf{r}_j)\times(\mathbf{p}_i-\mathbf{p}_j)\} \tag{1.1}$$

(iii) The tensor force

$$T(\mathbf{r}_{ij}) = V(\mathbf{r}_{ij})\{3(\boldsymbol{\sigma}_i\,.\,\mathbf{r}_{ij})(\boldsymbol{\sigma}_j\,.\,\mathbf{r}_{ij})/r_{ij}^2-(\boldsymbol{\sigma}_i\,.\,\boldsymbol{\sigma}_j)\} \tag{1.2}$$

and

(iv) The quadratic spin–orbit force

$$Q(\mathbf{r}_{ij}) = V(\mathbf{r}_{ij})\{(\boldsymbol{\sigma}_i\,.\,\boldsymbol{\sigma}_j)L_{ij}^2-Q_{ij}\} \tag{1.3}$$

where

$$\mathbf{L}_{ij} = (\mathbf{r}_i-\mathbf{r}_j)\times(\mathbf{p}_i-\mathbf{p}_j) \tag{1.4}$$

$$Q_{ij} = \tfrac{1}{2}\{(\boldsymbol{\sigma}_i\,.\,\mathbf{L}_{ij})(\boldsymbol{\sigma}_j\,.\,\mathbf{L}_{ij})+(\boldsymbol{\sigma}_j\,.\,\mathbf{L}_{ij})(\boldsymbol{\sigma}_i\,.\,\mathbf{L}_{ij})\}. \tag{1.5}$$

$V(\mathbf{r}_{ij})$ is a radial function, $\mathbf{s}_i$ is the Pauli spin matrix vector for the ith particle and $\mathbf{r}_i$ and $\mathbf{p}_i$ are its position and momentum vectors respectively.

The evidence for the existence of the non-central forces in the form of the linear spin–orbit and the tensor terms has been discussed by many authors (Signell and Marshak 1957, 1958, Gammel and Thaler 1957). The necessity for the inclusion of the quadratic spin–orbit term is confirmed by the work of Hamada (1960, 1961) and Hamada and Johnston (1962) in order to produce all the available data of p–p and n–p below 300 MeV.

In order to take care of the exchange nature of these forces between nucleons in various states, combinations of projection operators are constructed in such a way that each operator will pick out the pair of nucleons in a certain state only (Sribhibhadh 1966).

The projection operators are

$$\tfrac{1}{4}({}^{\nu}w_{\pm} + {}^{\nu}b_{\pm}B_{ij} + {}^{\nu}m_{\pm}M_{ij} + {}^{\nu}h_{\pm}H_{ij}) \tag{1.6}$$

ν being 1 and 3 to represent the singlet and the triplet states respectively, $\pm$ representing the even and the odd parity states. B_{ij}, M_{ij} and H_{ij} are the usual Bartlett, Majorana and Heisenberg exchange operators for the nuclei i and j.

The values of w, b, m and h are as follows:

$$ {}^{\nu}w_{\pm} = \quad 1 \text{ for all } \nu \tag{1.7}$$

$$\left.\begin{array}{llll} {}^{1}b_{+} = -1, & {}^{1}b_{-} = -1, & {}^{3}b_{+} = 1, & {}^{3}b_{-} = 1 \\ {}^{1}m_{+} = 1, & {}^{1}m_{-} = -1, & {}^{3}m_{+} = 1, & {}^{3}m_{-} = -1 \\ {}^{1}h_{+} = -1, & {}^{1}h_{-} = 1, & {}^{3}h_{+} = 1, & {}^{3}h_{-} = -1 \end{array}\right\} . \tag{1.8}$$

The potential shape functions $V(\boldsymbol{r}_{ij})$ in this work are taken to be Gaussian forms whose parameters are determined by least squares fits to the Hamada and Johnston potential (Hamada and Johnston 1962). Two Gaussian terms are used for each potential shape function so that one may represent the short-range contribution and the other the long-range one.

The interaction can now be written in full as

$$V_{ij} = \sum_{\lambda=1}^{4} \sum_{\nu=1}^{4} \tfrac{1}{4}(w + b_{\nu}B_{ij} + m_{\nu}M_{ij} + h_{\nu}H_{ij})\, {}^{\nu}V_{\lambda}(\boldsymbol{r}_{ij}, \boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j) + (e^2/r_{ij})\delta_{ij} \tag{1.9}$$

where

$$ {}^{\nu}V_{\lambda}(\boldsymbol{r}_{ij}, \boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j) = \sum_{\kappa=1}^{2} {}^{\lambda}U_{\kappa}^{\nu} \exp(-{}^{\lambda}\mu_{\kappa}^{\nu} r_{ij}^{2})[\delta_{\lambda,1} + \delta_{\lambda,2}\{\tfrac{1}{2}(\boldsymbol{\sigma}_i + \boldsymbol{\sigma}_j) \,.\, \boldsymbol{L}_{ij}\} $$
$$ + \delta_{\lambda,3}\{3(\boldsymbol{\sigma}_i \,.\, \boldsymbol{r}_{ij})(\boldsymbol{\sigma}_j \,.\, \boldsymbol{r}_{ij})/r_{ij}^{2} - (\boldsymbol{\sigma}_i \,.\, \boldsymbol{\sigma}_j)\} + \delta_{\lambda,4}\{(\boldsymbol{\sigma}_i \,.\, \boldsymbol{\sigma}_j)L_{ij}^{2} - Q_{ij}\}] \tag{1.10}$$

where e is the Coulomb electronic charge, $\delta_{ij} = 1$ if the nucleons are protons and zero otherwise, $\delta_{\lambda,\lambda'}$ is the kronecker delta having its usual meaning, λ represents the kth Gaussian term, λ takes the values 1, 2, 3 and 4 for the central, the linear spin–orbit, the tensor and the quadratic spin–orbit forces respectively. $\nu = 1, 2, 3$ and 4 to represent the triplet even state, the triplet odd state, the singlet even state and the singlet odd state respectively. The interaction given above is taken to be in the units in which

$$c = \hbar = 1. \tag{1.11}$$

2. The five-body wave function

In quantum mechanics the superposition of stationary states is used in molecular theory and Wheeler (1937) applying this to nuclear binding energy or scattering problems developed the method of the resonating group structures. This is the method we are going to apply to our five-body problem of n–^{4}He. This method specifies a wave function consisting of the product of the incompatible states and the sum of the compatible ones on a statistical basis.

Wheeler postulates that a given system of n nucleons subdivides into groups of stable nuclei and nucleons at any instant of time. Thus

$$\left.\begin{aligned}\Psi(123\ldots n) = {} & w_1\phi_1(\overset{1}{n}_1)\phi_2(\overset{1}{n}_2)\ldots\phi_\alpha(\overset{1}{n}_\alpha)+w_2\phi_1(\overset{2}{n}_1)\phi_2(\overset{2}{n}_2)\ldots\phi_\beta(\overset{2}{n}_\beta)+\ldots\\ & +w_\kappa\phi_1(\overset{\kappa}{n}_1)\phi_2(\overset{\kappa}{n}_2)\ldots\phi_\gamma(\overset{\kappa}{n}_\gamma)\end{aligned}\right\} \tag{2.1}$$

where for example $\phi_1(\overset{1}{n}_1)$ represents the wave function of a stable nucleus with n_1 nucleons and $\phi_2(\overset{1}{n}_2)$ the wave function of a stable nucleus with n_2 nucleons, etc., such that

$$\overset{1}{n}_1+\overset{1}{n}_2+\ldots+\overset{1}{n}_\alpha = n \qquad \text{etc.} \tag{2.2}$$

and w_1, w_2, etc. are the 'weights' proportional to the probability of each configuration (related to binding energy).

Consider a five-body problem, we can split the five-particle problem into deuterons, tritons and the alpha particles with their respective weights. However, since the alpha particle is strongly bound compared with deuterons and tritons we shall consider its contribution only.

The spatial coordinates in the five-body problem of n–^{4}He scattering are depicted in figure 1.

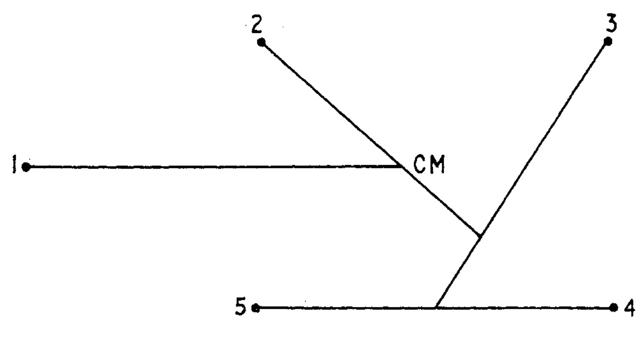

Figure 1. 1, 2 and 3 are neutrons and 4 and 5 are protons.

3. Neutron–alpha-particle scattering

Consider the problem of n–^{4}He scattering depicted in figure 2 symbolically. We have used the number 1 to represent the coordinate of the incident neutrons, 2 and 3 are neutrons in the helium nucleus and 4 and 5 the nuclear protons; 2, 3, 4 and 5 represents the alpha particle.

Since protons and neutrons are fermion particles, the wave function representing the motion of the system must be antisymmetric in 1, 2 and 3 and also in 4 and 5.

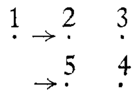

Figure 2

We may now write down the complete wave function in the resonating group using Hochberg's (1953) notation in the form:

$$\Psi = \psi(-1)\phi(1)+\psi(-2)\phi(2)+\psi(-3)\phi(3) \tag{3.1}$$

where $\psi(-i)$ is the antisymmetrized wave function of the target nucleus which does not contain the ith particle and ϕ is the wave function of the incident neutron.

The ground state of ^{4}He is of even parity and has a total angular momentum $J = 0$. The possible values of the orbital angular momentum L are 0, 1 and 2. Thus the ground state is a mixture of $^1S_{0-}$, $^3P_{0-}$ and $^5D_{0-}$ states (Gerjuoy and Schwinger 1942, Irving 1953 and Abraham *et al.* 1955). The principal $^3P_{0-}$ state will not be considered in this paper for its contribution is quite negligible for a nuclear force of the type we are going to consider. Similarly we do not take into account the $^3P_{0-}$ state probability because we have

$$\begin{aligned}\langle {}^1s_0|V_T|{}^3p_0\rangle &= \langle {}^1s_0|V_{LS}|{}^3p_0\rangle = \langle {}^3p_0|V_T|{}^5D_0\rangle \\ &= \langle {}^3p_0|V_{LS}|{}^5D_0\rangle = \langle {}^1s_0|V_{LS}|{}^5D_0\rangle \\ &= 0\end{aligned} \tag{3.2}$$

while only

$$\langle {}^1s_0|V_T|{}^5D_0\rangle \tag{3.3}$$

is non-zero.

Also, since the $^3p_{0-}$ state will appear only as a second approximation, it therefore follows that as a first approximation we can neglect the $^3p_{0-}$ state in our n–^{4}He scattering problems.

The alpha particle wave function $\psi(-1)$ can be written in the notation of Sugie *et al.* (1957) as

$$\begin{aligned}\psi(-1) &= \sum_{s} \psi_{L,s,0}(-1) = \sum_{s,p} \psi^{p}_{L,s,0}(-1) \\ &= \sum_{s,p} g^{p}_{s}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) w^{p}_{s}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}; \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_4)\chi(\widetilde{23}, \widetilde{45}).\end{aligned} \tag{3.4}$$

s can take two values namely $s = 0$ and $s = 2$. The superscript p distinguishes the many possible spin–angular wave functions; $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ are the three independent internal space coordinates and χ is the spin wave function of the singlet state.

We define χ as

$$\chi(\widetilde{23}, \widetilde{45}) = \tfrac{1}{2}(\alpha_2\beta_3-\beta_2\alpha_3)(\alpha_4\beta_5-\beta_4\alpha_5) \equiv \chi(-1). \tag{3.5}$$

We can now write equation (3.4) in the form

$$\psi(-1) = (\psi_S+c\psi_D)/(1+c^2)^{1/2}$$

where

$$\psi_S = g_S(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})\chi(-1) \tag{3.7}$$

and

$$\psi_D = g_D(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})w_D\chi(-1) \tag{3.8}$$

g_S and g_D represent the normalized spatial parts of the wave functions for the principal $^1s_{0-}$ and $^5D_{0-}$ state respectively.

w_D is defined by

$$w_D = \sum_{i>j=2}^{5} r_{ij}^2 s_{ij} = 3(\boldsymbol{\sigma}_2 \cdot \boldsymbol{r}_{23})(\boldsymbol{\sigma}_4 \cdot \boldsymbol{r}_{45}) + 3(\boldsymbol{\sigma}_2 \cdot \boldsymbol{r}_{45})(\boldsymbol{\sigma}_4 \cdot \boldsymbol{r}_{23}) - 2(\boldsymbol{\sigma}_2 \cdot \boldsymbol{\sigma}_4)(\boldsymbol{r}_{23} \cdot \boldsymbol{r}_{45}). \tag{3.9}$$

c^2 determines the amount of the D-state in the mixture. We assume that both ψ_S and ψ_D are normalized to unity, so that $\psi(-1)$ is then normalized to unity.

The radial parts ψ_S and ψ_D are of the form:

$$g_S = N_S \exp\left(-\tfrac{1}{2}\alpha \sum_{i>j=2}^{5} r_{ij}^2\right) \tag{3.10}$$

and

$$g_D = N_D \exp\left(-\tfrac{1}{2}\beta \sum_{i>j=2}^{5} r_{ij}^2\right) \tag{3.11}$$

where N_S and N_D are the normalization constants. The radial parts are assigned different variation parameters, α and β respectively.

The wave function of the incident neutron can be decomposed into partial waves by writing it in the form

$$\phi = \sum_{l,J=l\pm 1/2} \{f_{lJ}(r)/r\}\chi_{lJ}^{m}(\theta, \phi, s) \tag{3.12}$$

where (r, θ, ϕ) are the spherical polar coordinates of the neutron relative to the centre-of-mass (CM) of ^{4}He and χ_{lJ}^m is the spin–angular wave function of the neutrons. Since the spin of ^{4}He is zero, it follows therefore that the magnetic quantum number m is also the magnetic quantum number of the whole system.

Using the notation of Buckingham and Massey (1941), the alpha-particle wave function can be expanded in the eigenfunctions of J and M as follows:

$$\Psi_M = \sum_{\text{cyclic } 1,2,3} \sum_{J=1/2}^{\infty} \sum_{l=J-1/2}^{J+1/2} \frac{1}{r_1} f_{JMl}(r_1) \times \{C_{JMl,M+1/2}\mathscr{Y}_{l,M+1/2}(\theta_1, \phi_1)\beta_1 + C_{JMl,M-1/2}\mathscr{Y}_{l,M-1/2}(\theta_1, \phi_1)\alpha_1\}\psi(-1)\chi(-1) \tag{3.13}$$

$M\hbar$ being the z-component of the total angular momentum, C_{JMlm} the appropriate Clebsch–Gordon coefficients and the $\mathscr{Y}_{l,m}$ spherical harmonics. Using the notation of Hochberg *et al.* (1968—unpublished), each of the eigenfunctions may be expressed in terms of sub-states characterized by the quantum numbers l and s of the orbital angular momentum and the spin. Thus, we write the complete wave function as

$$\Psi_M = \sum_{s} \psi(-1)\mathscr{F}_{JM}^{s}F(1-2345) \tag{3.14}$$

where

$$\mathscr{F}^s_{JM}(1-2345) = \sum_{l=|J-s|}^{J+s} \mathscr{Y}_{JMl}(1, \overline{2345}; \Omega)\frac{1}{r}f^s_{JMl}(r) \quad (3.15)$$

$$\mathscr{Y}^s_{JMl}(1, \overline{2345}; \Omega) = \sum_{m=M-s}^{M+s} C^s_{JlMm}\mathscr{Y}_{lM}(\Omega)\chi^s_{M-m}(1, \overline{2345}) \quad (3.16)$$

Ω being the solid angle of $\boldsymbol{r}$.

For each J and M, in the doublet spin state ($s = \frac{1}{2}$) for example, we put

$$\chi^{1/2}(1) = \left\{\chi^{1/2}_{1/2}(1, \overline{2345}; \chi^{1/2}_{-1/2}(1, \overline{2345})\right\} \quad (3.17)$$

$$\mathscr{Y}^{1/2}_{JMl}(\Omega) = \begin{Bmatrix} C^{1/2}_{JM(J-1/2)(M-1/2)}\mathscr{Y}^{J-1/2}_{M-1/2}(\theta, \phi), & C^{1/2}_{JM(J+1/2)(M-1/2)}\mathscr{Y}^{J+1/2}_{M-1/2}(\theta, \phi) \\ C^{1/2}_{JM(J-1/2)(M+1/2)}\mathscr{Y}^{J-1/2}_{M+1/2}(\theta, \phi), & C^{1/2}_{JM(J+1/2)(M+1/2)}\mathscr{Y}^{J+1/2}_{M+1/2}(\theta, \phi) \end{Bmatrix} \quad (3.18)$$

and

$$\mathscr{F}_{JM}(r) = \begin{Bmatrix} f^{1/2}_{JMJ-1/2}(r) \\ f^{1/2}_{JMJ+1/2}(r) \end{Bmatrix}. \quad (3.19)$$

Equation (3.18) is a 2×2 matrix. Thus equation (3.15) can now be rewritten in the form

$$\mathscr{F}^s_{JM}(1-2345) = \frac{1}{r}\chi\mathscr{Y}\mathscr{F}. \quad (3.20)$$

Here $\mathscr{Y}$ always refers to $\mathscr{Y}^s_{JMl}(\Omega)$ which involves no spin functions whereas $\mathscr{Y}^s_{JMl}(1, \overline{2345}; \Omega)$ does, as shown by its argument, and will always be written in full.

4. Formulation of n–^{4}He scattering problems

The formulation we are going to use is based on I. We start with the basic Schrödinger equation for the wave function ψ given by

$$\left(T_{12345} + \sum_{i>j=1}^{5} V_{ij}\right)\Psi(\widetilde{123}, \widetilde{45}) = E\Psi(\widetilde{123}, \widetilde{45}) \quad (4.1)$$

where E is the total energy of the system in the CM system. E is given by

$$E = E_n + E_\alpha \quad (4.2)$$

where E_n is the energy of the incident neutron, and E_α is the energy of the alpha ground state, i.e. the binding energy of the alpha particle. T_{12345} is the kinetic energy operator and V_{12345} the potential energy.

Following the same procedures as given in I, we can write equation (4.1) using equation (1.2.3.) in I as:

$$\begin{aligned}(T_{1-2345} - E_n)\mathscr{F}(1-2345) + 4\int d\tau_{2345}\phi^*(1, 2345)V_{12}\phi(1, 2345)\mathscr{F}(1-2345) \\ -2\int d\tau_{2345}\phi^*(1, 2345)\Big(\tfrac{1}{2}T_{1-2345} + \tfrac{1}{2}T_{2-1345} + V_{12} - E_n \\ -\tfrac{1}{2}T_{1345} - \tfrac{1}{2}T_{2345} + E_\alpha - 3V_{34} - \frac{e^2}{r_{45}}\Big)\phi(2-1345)\mathscr{F}(2-1345) = 0.\end{aligned} \quad (4.3)$$

The equation for $f_{lJ}(r)$ is given by

$$\frac{\hbar^2}{2M}\left(\frac{d^2}{dr^2}+k^2-\frac{l(l+1)}{r^2}\right)f_{lJ}(r) = 4\int d\tau_{-1}\, d\Omega_1 \psi^*(-1)\bar{\chi}_{lJ}^{m}(1)V_{12}\psi(-1)$$
$$\times \chi_{lJ}^{m}(1)f_{lJ}(r) - 2\int d\tau_{-1}\, d\Omega_1\psi^*(-1)\bar{\chi}_{lJ}^{m}(1)$$
$$\times\left(V_{12}-3V_{34}-\frac{e^2}{r_{45}}\right)\psi(-2)\chi_{lJ}^{m}(2)\frac{f_{lJ}(r')r}{r'}$$
$$-2\int d\tau_{-1}\, d\Omega_1\psi^*(-1)\bar{\chi}_{lJ}^{m}(1)(E_\alpha-k^2)\psi(-2)\chi_{lJ}^{m}(2)$$
$$\times\frac{f_{lJ}(r')r}{r'}+2\frac{\hbar^2}{2M}\int d\tau_{-1}\, d\Omega_1\psi^*(-1)\bar{\chi}_{lJ}^{m}(1)$$
$$\times\left(\tfrac{1}{2}\nabla^2_{1-2345}+\tfrac{1}{2}\nabla^2_{2-1345}-\tfrac{1}{2}\nabla^2_{2345}-\tfrac{1}{2}\nabla^2_{1345}\right)$$
$$\times\psi(-2)\chi_{lJ}^{m}(2)\frac{f_{lJ}(r')r}{r'} \tag{4.4}$$

where

$$k^2 = \frac{8M}{5\hbar^2}E_n \tag{4.5}$$

$$\left.\begin{aligned} \boldsymbol{r} &= \boldsymbol{r}_1-\tfrac{1}{4}(\boldsymbol{r}_2+\boldsymbol{r}_3+\boldsymbol{r}_4+\boldsymbol{r}_5)\\ \boldsymbol{r}' &= \boldsymbol{r}_2-\tfrac{1}{4}(\boldsymbol{r}_1+\boldsymbol{r}_3+\boldsymbol{r}_4+\boldsymbol{r}_5)\end{aligned}\right\}. \tag{4.6}$$

5. The complete integro-differential equations for the elastic scattering of n–^{4}He

Coupled equations do not arise in the analysis of the scattering of n–^{4}He. We shall therefore write down the full integro-differential equations for the uncoupled states for the radial wave functions. We shall collect the results obtained in equation (4.4) using equation (1.9) and regroup the terms in a way convenient for numerical calculations.

Below are the complete integro–differential equations (for both the S- and D-state alpha wave functions) including the linear spin–orbit interaction, tensor interaction and the quadratic spin–orbit interaction.

$$\mathscr{D}_l^2 f_{lJ}(r) = \mathscr{T}_l(r)\frac{1}{r^2}f_{lJ}(r)+\mathscr{U}_l(r)f_{lJ}(r)+\mathscr{V}_l(r)\frac{1}{r}\frac{d}{dr}f_{lJ}(r)+\mathscr{W}_l(r)\frac{d^2}{dr^2}f_{lJ}(r)$$
$$+\int_0^\infty\{\mathscr{N}_l(r,r')+\mathscr{P}_l(r,r')+\mathscr{S}_l(r,r')+\mathscr{K}_l(r,r')\}f_{lJ}(r')\,dr' \tag{5.1}$$

where

$$l = J+\tfrac{1}{2} \qquad \text{and} \qquad l = J-\tfrac{1}{2} \tag{5.2}$$

$$\left.\begin{aligned} \mathscr{D}_l^2 &= \frac{d^2}{dr^2}+K^2-\frac{l(l+1)}{r^2}\\ K^2 &= \frac{8M}{5\hbar^2}E_n\end{aligned}\right\} \tag{5.3}$$

and

$$\begin{aligned}
\mathscr{T}_l(r) &= {}^{Q}\mathscr{T}_l(r)\\
\mathscr{U}_l(r) &= {}^{c}\mathscr{U}(r) + {}^{Ls}\mathscr{U}_l(r) + {}^{\mathrm{Coul}}\mathscr{U}(r)\\
\mathscr{V}_l(r) &= {}^{Q}\mathscr{V}(r)\\
\mathscr{W}_l(r) &= {}^{Q}\mathscr{W}(r)\\
\mathscr{K}_l(r,r') &= {}^{c}\mathscr{K}_l(r,r') + {}^{Ls}\mathscr{K}_l(r,r') + {}^{Q}\mathscr{K}_l(r,r') + {}^{\mathrm{Coul}}\mathscr{K}_l(r,r') + {}^{T}\mathscr{K}_l'(r,r')\\
&\quad + {}^{T}\mathscr{K}_l''(r,r').
\end{aligned} \tag{5.4}$$

The above terms represent the contribution from various interactions indicated by the symbols attached. The prime and the double prime symbols denote the S–D and the D–S interactions respectively. In writing down the S–D and D–S terms we shall omit the indices and the prime symbols where there is no confusion.†

5.1. *The* S–S *direct terms*

5.1.1. *The potential terms*

$${}^{c}\mathscr{U}(r) = \frac{8M}{5\hbar^2}\sum_{\nu=1}(4w+2b-m-2h)\nu \int V(|\boldsymbol{r}-\tfrac{3}{4}\boldsymbol{\rho}_2|)|\phi(\boldsymbol{\rho}_2,\boldsymbol{\rho}_3,\boldsymbol{\rho}_4)|^2\,\mathrm{d}\boldsymbol{\rho}_2\,\mathrm{d}\boldsymbol{\rho}_3\,\mathrm{d}\boldsymbol{\rho}_4$$

where

$$V(|\boldsymbol{r}-\tfrac{3}{4}\boldsymbol{\rho}_2|) \equiv U(r_{12}) = \sum_{\kappa=1}^{2} {}^{c}U_\kappa^\nu \exp(-{}^{c}\mu_\kappa^\nu r_{12}^2)$$

and

$$|\phi(\boldsymbol{\rho}_2,\boldsymbol{\rho}_3,\boldsymbol{\rho}_4)|^2 = \frac{N_S^2}{1+c^2}\exp\left\{-\alpha\left(3\rho_2^2+\frac{8\rho_3^2}{3}+2\rho_4^2\right)\right\} \equiv |\varphi(-1)|^2$$

with

$$N_S^2 = \left(\frac{16\alpha^3}{\pi^3}\right)^{3/2}.$$

On performing the integrations over complete vectors, the above integral leads to the following:

$${}^{c}\mathscr{U}(r) = \frac{8M}{5\hbar^2}\sum_{\kappa=1}^{2}\sum_{\nu=1}^{4}(4w+2b-m-2h)\nu\frac{{}^{c}U_\kappa^\nu}{1+c^2}P_c^{3/2}(\kappa)\exp\{-P_c(\kappa)\,{}^{c}\mu_\kappa^\nu r^2\}$$

$${}^{Q}\mathscr{T}_l(r) + {}^{Q}\mathscr{U}_l(r) + {}^{Q}\mathscr{V}(r) + {}^{Q}\mathscr{W}(r) = \frac{8M}{5\hbar^2}r\int \mathrm{d}s\,\mathrm{d}\Omega\{\phi(-1)\chi(-1)\mathscr{Y}(\Omega)\}^*$$

$$\times \sum_{\nu} 2(m-2b)\nu\,{}^{\nu}V_{LL}(12)Qs(12)\phi(-1)\chi(-1)\mathscr{Y}(\Omega)\mathscr{F}(r)$$

where

$$\mathscr{F}(r) = \frac{1}{r}f(r)$$

and

$$Qs(12) = (\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)L_{12}^2 - Q_{12}$$

$$Q_{12} = \tfrac{1}{2}\{(\boldsymbol{\sigma}_1\cdot\boldsymbol{L}_{12})(\boldsymbol{\sigma}_2\cdot\boldsymbol{L}_{12}) + (\boldsymbol{\sigma}_2\cdot\boldsymbol{L}_{12})(\boldsymbol{\sigma}_1\cdot\boldsymbol{L}_{12})\}$$

$$\mathrm{i}\boldsymbol{L}_{12} = (\boldsymbol{r}_1-\boldsymbol{r}_2)\times\left(\frac{\partial}{\partial\boldsymbol{r}_1}-\frac{\partial}{\partial\boldsymbol{r}_2}\right) = (\boldsymbol{r}-\tfrac{3}{4}\boldsymbol{\rho}_2)\times\left(\frac{5}{4}\frac{\partial}{\partial\boldsymbol{r}}-\frac{\partial}{\partial\boldsymbol{\rho}_2}\right).$$

† Note that the symbols C, LS, T and LL or Q represent the central, linear spin–orbit, tensor and quadratic spin–orbit force respectively and Coul the Coulomb energy term.

On using the following relations:

$$\boldsymbol{\sigma}_1\chi(\widetilde{13}) = -\boldsymbol{\sigma}_3\chi(\widetilde{13}), \qquad \boldsymbol{\sigma}_2\chi(\widetilde{23}) = -\boldsymbol{\sigma}_3\chi(\widetilde{23}), \qquad (\boldsymbol{\sigma}.\boldsymbol{\sigma}) = 3$$

and

$$L^2 r\mathscr{Y}\mathscr{F} = \frac{l(l+1)}{r} f\mathscr{Y}$$

$$\left(\boldsymbol{r}\cdot\frac{\partial}{\partial \boldsymbol{r}}\right)\mathscr{Y}\mathscr{F} = \left(f' - \frac{1}{r}f\right)\mathscr{Y}$$

$$\left(\frac{\partial}{\partial \boldsymbol{r}}\right)^2 \mathscr{Y}\mathscr{F} = \frac{1}{r}\left\{f'' - \frac{l(l+1)}{r^2}\right\}\mathscr{Y}$$

and on performing the integrations over complete vectors, the above integral becomes

$$-\frac{8M}{5\hbar^2}\sum_{\kappa=1}^{2}\sum_{\nu=1}^{4} 2(m-2b)\nu \frac{{}^{LL}U_\kappa^\nu}{1+c^2} P_{LL}^{3/2}(\kappa)\exp\{-P_{LL}(\kappa){}^{LL}\mu_\kappa^\nu r^2\}$$
$$\times\left[\left\{\left(\frac{25}{16}(1-\tfrac{3}{4}\gamma)^2 + \left(\frac{15}{16}\right)^2\frac{1}{\lambda r^2}\right)l(l+1) - \left(\frac{135\alpha}{16\lambda} + 2\sigma r^2 + 2\eta\right)\right\}f_{lJ}(r)\right.$$
$$\left. + 2\eta r f'_J(r) - \left(\frac{15}{16}\right)^2\frac{1}{\lambda}f''_{lJ}(r)\right]$$

that is

$${}^{Q}\mathscr{T}_l(r) = -\left(\frac{16}{15}\right)^2\frac{8M}{5\hbar^2}\sum_{\kappa=1}^{2}\sum_{\nu=1}^{4} 2(m-2b)\nu\frac{{}^{LL}U_\kappa^\nu}{1+c^2}\frac{l(l+1)}{\lambda_{LL}(\kappa)}P_{LL}^{3/2}(\kappa)\exp\{-P_{LL}(\kappa){}^{LL}\mu_\kappa^\nu r^2\}$$

and

$${}^{Q}\mathscr{U}_l(r) = \frac{8M}{5\hbar^2}\sum_{\kappa=1}^{2}\sum_{\nu=1}^{4} 2(m-2b)\nu\frac{{}^{LL}U_\kappa^\nu}{1+c^2}P_{LL}^{3/2}(\kappa)\left[\frac{135\alpha}{16\lambda_{LL}(\kappa)} - \frac{25}{16}\left\{1-\frac{3}{4}\gamma_{LL}(\kappa)\right\}^2 l(l+1)\right.$$
$$\left. + 2\sigma r^2 + 2\eta\right]\exp\{-P_{LL}(\kappa){}^{LL}\mu_\kappa^\nu r^2\}$$

and

$${}^{Q}\mathscr{V}(r) = -\frac{8M}{5\hbar^2}\sum_{\kappa=1}^{2}\sum_{\nu=1}^{4} 2(m-2b)\nu\frac{{}^{LL}U_\kappa^\nu}{1+c^2}P_{LL}^{3/2}(\kappa)(2\eta r^2)\exp\{-P_{LL}(\kappa){}^{LL}\mu_\kappa^\nu r^2\}$$

and

$${}^{Q}\mathscr{W}(r) = \left(\frac{16}{15}\right)^2\frac{8M}{5\hbar^2}\sum_{\kappa=1}^{2}\sum_{\nu=1}^{4} 2(m-2b)\nu\frac{{}^{LL}U_\kappa^\nu}{1+c^2}\left\{\frac{1}{\lambda_{LL}(\kappa)}\right\}P_{LL}^{3/2}(\kappa)\exp\{-P_{LL}(\kappa){}^{LL}\mu_\kappa^\nu r^2\}$$

$${}^{Ls}\mathscr{U}_l(r) = \frac{8M}{5\hbar^2}\int \mathrm{d}r\,\mathrm{d}\Omega\{\phi(-1)\chi(-1)\mathscr{Y}(\Omega)\}^*$$
$$\times\sum_\nu (w-\tfrac{1}{2}m)\nu\,{}^\nu V_{Ls}(12)\phi(-1)\chi(-1)\mathscr{Y}(\Omega)f_{lJ}(r).$$

The potential in this case is

$$\sum_\nu\sum_\kappa {}^{Ls}U_\kappa^\nu\exp(-{}^{Ls}\mu_\kappa^\nu r_{12}^2) = \sum_\nu (w-\tfrac{1}{2}m)\nu\,{}^\nu V_{Ls}(12).$$

On writing the spin–orbit operator $V_{Ls}(12)$ in full, the above integral becomes

$$
{}^{Ls}\mathscr{U}_l(r) = \frac{8M}{5\hbar^2} N_S^2 \frac{1}{2\mathrm{i}} \sum_{\kappa=1}^{2} \sum_{\nu=3}^{4} 4\left(w - \frac{m}{2}\right)\nu \frac{{}^{Ls}U_\kappa^\nu}{1+c^2} \int \exp\left\{-\left(\frac{8}{3}\alpha\rho_3^2 + 2\alpha\rho_4^2\right)\right\} \mathrm{d}\boldsymbol{\rho}_3\, \mathrm{d}\boldsymbol{\rho}_4
$$

$$
\times \int \exp\left\{-\frac{3\alpha\rho_2^2}{2} - {}^{Ls}\mu_\kappa^\nu(\boldsymbol{r} - \tfrac{3}{4}\boldsymbol{\rho}_2)^2\right\} \bar{\chi}(-1)\bar{\chi}_{lJ}^m(1)
$$

$$
\times \left[(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \cdot \left\{(\boldsymbol{r} - \tfrac{3}{4}\boldsymbol{\rho}_2) \times \left(\frac{5}{4}\frac{\partial}{\mathrm{d}\boldsymbol{r}} - \frac{\partial}{\partial\boldsymbol{\rho}_2}\right)\right\}\right]
$$

$$
\times \exp\left\{-\frac{3\alpha\rho_2^2}{2}\right\} \chi_{lJ}^m(1)\chi(-1)\, \mathrm{d}\boldsymbol{r}\, \mathrm{d}\boldsymbol{\rho}_2\, \mathrm{d}\Omega.
$$

Using the relation that

$$
\sum \chi^*(-1)\boldsymbol{\sigma}_2\chi(-1) = 0, \qquad \text{and} \qquad \mathrm{i}\boldsymbol{\sigma} \cdot \boldsymbol{r} \times \frac{\partial}{\partial \boldsymbol{r}} = \boldsymbol{L} \cdot \boldsymbol{\sigma}
$$

the last integral becomes:

$$
{}^{Ls}\mathscr{U}_l(r) = \frac{8M}{5\hbar^2} \sum_{\kappa=1}^{2} \sum_{\nu=3}^{4} (w - \tfrac{1}{2}m)\nu \frac{{}^{Ls}U_\kappa^\nu}{1+c^2} \frac{5}{2} \|\boldsymbol{\sigma}_l\| P_{Ls}^{3/2}(\kappa) \exp\{-P_{Ls}(\kappa)^{Ls}\mu_\kappa^\nu r^2\}
$$

$$
{}^{\mathrm{Coul}}\mathscr{U}(r) = \frac{8M}{5\hbar^2} e^2 \int |\phi(\boldsymbol{\rho}_2, \boldsymbol{\rho}_3, \boldsymbol{\rho}_4)|^2 \frac{1}{\boldsymbol{\rho}_4} \mathrm{d}\boldsymbol{\rho}_2\, \mathrm{d}\boldsymbol{\rho}_3\, \mathrm{d}\boldsymbol{\rho}_4
$$

where

$$
|\phi(\boldsymbol{\rho}_2, \boldsymbol{\rho}_3, \boldsymbol{\rho}_4)|^2 = \frac{N_S^2}{1+c^2} \exp\left\{-\alpha\left(3\rho_2^2 + \frac{8}{3}\rho_3^2 + 2\rho_4^2\right)\right\}.
$$

The above integral is easily evaluated and the result is

$$
{}^{\mathrm{Coul}}\mathscr{U}(r) = \frac{16M}{5\hbar^2} e^2 \frac{1}{1+c^2} \left(\frac{2\alpha}{\pi}\right)^{1/2}.
$$

5.2. *The* S–S *indirect terms*

5.2.1. *The energy terms*

$$
\mathscr{N}_{s-s}(r) = -2\left(\frac{16}{15}\right)^3 r \frac{8M}{5\hbar^2} (E_n + E_\alpha) \int \mathrm{d}\boldsymbol{r}'\, \mathscr{Y}^*(\Omega)\chi^*(-1)\phi(\boldsymbol{s}', \boldsymbol{\rho}_3, \boldsymbol{\rho}_4)\phi(\boldsymbol{s}, \boldsymbol{\rho}_3, \boldsymbol{\rho}_4')
$$

$$
\times \chi(-2)\mathscr{Y}(\Omega') \frac{1}{r'} f_{lJ}(r')\, \mathrm{d}\boldsymbol{\rho}_3\, \mathrm{d}\boldsymbol{\rho}_4
$$

where

$$
\boldsymbol{s} = \frac{4}{15}(4\boldsymbol{r} + \boldsymbol{r}'), \qquad \boldsymbol{s}' = \frac{4}{15}(\boldsymbol{r} + 4\boldsymbol{r}')
$$

and

$$
\chi^*(-1)\chi(-2) = -\tfrac{1}{2}.
$$

The above kernel is easily evaluated and the final result is given by

$$\mathcal{N}_{s-s}(r) = \frac{1}{1+c^2}\frac{128}{15}\left(\frac{3\alpha}{\pi}\right)^{1/2}(E_n+E_\alpha)\int \exp\left\{-\frac{136\alpha}{75}(r^2+r'^2)\right\}\mathscr{B}_{l+1/2}(\epsilon rr')f_{lJ}(r')\mathrm{d}r'.$$

Hence

$$\mathcal{N}_l(r,r') = \frac{128}{15}\frac{1}{1+c^2}\left(\frac{3\alpha}{\pi}\right)^{1/2}\frac{8M}{5\hbar^2}(E_n+E_\alpha)\exp\left\{-\frac{17}{16}\epsilon(r^2+r'^2)\right\}\mathscr{B}_{l+1/2}(\epsilon rr').$$

$$\mathscr{S}_{s-s}(r) = -2r\int \phi(-1)\left(\nabla^2_{1-2345}+\frac{16}{15}\nabla^2_{2-345}+\frac{6}{5}\nabla^2_{3-45}+\frac{8}{5}\nabla^2_{45}\right)$$

$$\times\phi(-2)\frac{1}{r'}f_{lJ}(r')\,\mathrm{d}\mathbf{r}'.$$

On performing the integrations of the above integral over complete vectors and over spherical harmonics, we arrive at the following result:

$$\mathscr{S}_l(r,r') = \frac{1}{1+c^2}\left(\frac{16}{15}\right)\left(\frac{3\alpha}{\pi}\right)^{1/2}\epsilon^2\exp\left\{-\frac{17}{16}\epsilon(r^2+r'^2)\right\}$$

$$\times\left[\left\{38(r^2+r'^2)-\frac{1}{\epsilon}(49l+141)\right\}\mathscr{B}_{l+1/2}(\epsilon rr')+49rr'\mathscr{B}_{l+3/2}(\epsilon rr')\right]$$

$$\mathscr{P}_{s-s}(r) = -\left(\frac{16}{15}\right)^3\frac{8Me^2}{5\hbar^2}\frac{N_S^2}{1+c^2}\int\frac{1}{\rho_4}\exp\left\{-\frac{3\alpha}{2}(s'^2+s^2)-\frac{8\alpha\rho_3^2}{3}-2\alpha\rho_4^2\right\}$$

$$\times f_{lJ}(r')rr'\,\mathrm{d}r'\,\mathrm{d}\boldsymbol{\rho}_3\,\mathrm{d}\boldsymbol{\rho}_4$$

$$= -\frac{1}{1+c^2}\left(\frac{3\alpha}{\pi}\right)^{1/2}\frac{32}{15}\epsilon^2\left\{\frac{4M'}{\epsilon}\left(\frac{2\alpha}{\pi}\right)^{1/2}e^2\right\}\int\exp\left\{-\frac{17}{16}\epsilon(r^2+r'^2)\right\}$$

$$\times\mathscr{B}_{l+1/2}(\epsilon rr')f_{lJ}(r')\,\mathrm{d}r'$$

where

$$M' = \frac{15}{8\alpha}\frac{M}{\hbar^2}\qquad\text{and}\qquad \epsilon = \frac{128\alpha}{75}.$$

Hence

$$\mathscr{P}_l(r,r') = -\frac{e^2}{1+c^2}\left(\frac{3\alpha}{\pi}\right)^{1/2}\left(\frac{2\alpha}{\pi}\right)^{1/2}\frac{16M}{\alpha\hbar^2}\epsilon\exp\left\{-\frac{17}{16}\epsilon(r^2+r'^2)\right\}\mathscr{B}_{l+1/2}(\epsilon rr').$$

5.2.2. *The potential terms*

$${}^c\mathscr{K}_l(r,r') = {}^c\mathscr{K}_l^{(1)}(r,r')+{}^c\mathscr{K}_l^{(2)}(r,r')+{}^c\mathscr{K}_l^{(3)}(r,r')$$

where

$${}^c\mathscr{K}_l^{(1)}(r,r') = \frac{N_S^2}{1+c^2}\left(\frac{16}{15}\right)^3\frac{8M}{5\hbar^2}\sum_{\kappa=1}^{2}\sum_{\nu=1}^{4}(4m+2h-w-2b)_\nu{}^cU_\kappa^\nu$$

$$\times\int\exp\left\{-\frac{3\alpha}{2}(s'^2+s^2)-\frac{8\alpha\rho_3^2}{3}-2\alpha\rho_4^2-{}^c\mu_\kappa^\nu(s'-s)^2\right\}\mathrm{d}\boldsymbol{\rho}_3\,\mathrm{d}\boldsymbol{\rho}_4$$

$$= \left(\frac{16}{15}\right)^3 \left(\frac{3\alpha}{\pi}\right)^{3/2} \frac{M}{\hbar^2} \sum_{\kappa=1}^{2} \sum_{\nu=1}^{4} (4m+2h-w-2b)\nu \frac{{}^cU_\kappa^\nu}{1+c^2}\left(\frac{15\pi}{4\alpha-3{}^c\mu_\kappa^\nu}\right)$$

$$\times \exp\left\{-\frac{8}{75}(17\alpha+6{}^c\mu_\kappa^\nu)(r^2+r'^2)\right\}\mathscr{B}_{l+1/2}\{\lambda_c(\kappa)rr'\}$$

$${}^c\mathscr{K}_l^{(2)}(r,r') = \frac{N_S^2}{1+c^2}\left(\frac{16}{15}\right)^3 \frac{8M}{25\hbar^2}\int \exp\left\{-\frac{3\alpha}{2}(s'^2+s^2)-\frac{8\alpha\rho_3^2}{3}-2\alpha\rho_4^2\right\}$$

$$\times V(|\mathbf{s}'-\tfrac{2}{3}\boldsymbol{\rho}_3|)\,d\boldsymbol{\rho}_3\,d\boldsymbol{\rho}_4 + \text{ a similar term with } r \rightleftharpoons r'.$$

where

$$V(|\mathbf{s}'-\tfrac{2}{3}\boldsymbol{\rho}_3|) = \sum_{\kappa=1}^{2}\sum_{\nu=1}^{4}(w+m)\nu\,{}^cU_\kappa^\nu \exp\{-{}^c\mu_\kappa^\nu(s'-\tfrac{2}{3}\boldsymbol{\rho}_3)^2\}.$$

The integrations of the above integral over complete vectors and over spherical harmonics lead to:

$${}^c\mathscr{K}_l^{(2)}(r,r') = -3\left(\frac{16}{15}\right)^3\left(\frac{3\alpha}{\pi}\right)^{3/2}\frac{M}{\hbar^2}\sum_{\kappa=1}^{2}\sum_{\nu=1}^{4}(w+m)\nu\frac{{}^cU_\kappa^\nu}{1+c^2}\frac{3\pi}{2(2\alpha+{}^c\mu_\kappa^\nu)}$$

$$\times\eta^{1/2}(\kappa)\left[\exp\left\{-\frac{4}{75}\eta(\kappa)((34\alpha+7\,{}^c\mu_\kappa^\nu)r^2+(34\alpha+27\,{}^c\mu_\kappa^\nu)r'^2)\right\}\right.$$

$$\left.\times\,\mathscr{B}_{l+1/2}\{\xi(\kappa)rr'\}+\text{a similar term with } r\rightleftharpoons r'.\right]$$

$${}^c\mathscr{K}_l^{(3)}(r,r') = 17\frac{N_S^2}{1+c^2}\left(\frac{16}{15}\right)^3\frac{8M}{25\hbar^2}\sum_{\kappa=1}^{2}\sum_{\nu=1}^{4}(w+m)\nu\,{}^cU_\kappa^\nu$$

$$\times\int\exp\left\{-\tfrac{3}{2}\alpha(s'^2+s^2)-\frac{8\alpha\rho_3^2}{3}-(2\alpha+{}^c\mu_\kappa^\nu)\rho_4^2\right\}d\boldsymbol{\rho}_3\,d\boldsymbol{\rho}_4$$

$$= -3\left(\frac{16}{15}\right)^3\left(\frac{3\alpha}{\pi}\right)^{3/2}\frac{M}{\hbar^2}\sum_{\kappa=1}^{2}\sum_{\nu=1}^{4}(w+m)\nu\frac{{}^cU_\kappa^\nu}{1+c^2}\frac{51\pi}{4\alpha}\left(\frac{2\alpha}{2\alpha+{}^c\mu_\kappa^\nu}\right)^{3/2}$$

$$\times\exp\left\{-\frac{17}{16}\epsilon(r^2+r'^2)\right\}\mathscr{B}_{l+1/2}(\epsilon rr')$$

$${}^{Ls}\mathscr{K}_l(r,r') = \frac{N^2s}{1+c^2}\frac{8M}{5\hbar^2}\frac{1}{2i}\left(\frac{16}{15}\right)^3\sum_{\kappa=1}^{2}\sum_{\nu=3}^{4}2(w-2m)\nu\,{}^{Ls}U_\kappa^\nu\int\chi^*(-1)\bar{\chi}_{lJ}^{m}(1)$$

$$\times\exp\left\{-\frac{3\alpha s'^2}{2}-\frac{16}{15}{}^{Ls}\mu_\kappa^\nu(\mathbf{r}-\mathbf{r}')-\alpha(\tfrac{4}{3}\rho_3^2+\rho_4^2)\right\}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)$$

$$\cdot\left\{(\mathbf{r}-\mathbf{r}')\times\left(\frac{\partial}{\partial\mathbf{r}}-\frac{\partial}{\partial\mathbf{r}'}\right)\right\}\exp\left\{-\frac{3}{2}\alpha s^2-\alpha\left(\frac{4}{3}\rho_3^2+\rho_4^2\right)\right\}\chi_{lJ}^m(2)\chi(-2)f_{lJ}(r')$$

$$\times rr'\,dr'\,d\boldsymbol{\rho}_3\,d\boldsymbol{\rho}_4\,d\Omega'.$$

On using Green's theorem and the following relation

$$\int f(r')\,\mathrm{d}r'\,\mathrm{d}\Omega'\,\boldsymbol{r}\times\boldsymbol{r}'\exp\{-\delta(r^2+r'^2)-\epsilon\boldsymbol{r}\,.\,\boldsymbol{r}'\}$$
$$= -\frac{1}{\epsilon}\int f(r')\,\mathrm{d}r'\,\mathrm{d}\Omega'\,\boldsymbol{r}'\times\frac{\partial}{\partial\boldsymbol{r}'}\exp\{-\delta(r^2+r'^2)-\epsilon\boldsymbol{r}\,.\,\boldsymbol{r}'\}$$

and performing integrations over complete vectors and over spherical harmonics, we arrive at the following final result:

$$^{Ls}\mathscr{K}_l(r,r') = -\left(\frac{16}{15}\right)^3\left(\frac{3\alpha}{\pi}\right)^{3/2}\frac{M}{\hbar^2}\sum_{\kappa=1}^{2}\sum_{\nu=3}^{4}(w-2m)\nu\frac{^{Ls}U_\kappa^\nu}{1+c^2}$$
$$\times\left(\frac{20\alpha}{4\alpha-3^{Ls}\mu_\kappa^\nu}\right)\left(\frac{15\pi}{4\alpha-3^{Ls}\mu_\kappa^\nu}\right)\|\boldsymbol{\sigma}_l\|\exp\left\{-\frac{8}{75}(17\alpha+6^{Ls}\mu_\kappa^\nu)(r^2+r'^2)\right\}$$
$$\times\mathscr{B}_{l+1/2}\{\lambda_{Ls}(\kappa)rr'\}$$

$$^{Q}\mathscr{K}_{s-s}(r,r') = \left(\frac{16}{15}\right)^3\frac{8M}{5\hbar^2}N_S^2 r\sum_{\kappa=1}^{2}\sum_{\nu=1}^{4}2(w-2h)\nu\frac{^{LL}U_\kappa^\nu}{1+c^2}\int\chi^*(-1)\mathscr{Y}^*_{JlM}(1)$$
$$\times\exp\left[-\frac{\alpha}{2}\left(3s'^2+\frac{8\rho_3^2}{3}+2\rho_4^2\right)-{}^{LL}\mu_\kappa^\nu\left\{\frac{4}{5}(\boldsymbol{r}-\boldsymbol{r}')\right\}^2\right]$$
$$\times Q_s(12)\sum_{J,l}\mathscr{Y}_{JlM}(2)\chi(-2)\exp\left\{-\frac{\alpha}{2}\left(3s^2+\frac{8\rho_3^2}{3}+2\rho_4^2\right)\right\}$$
$$\times\frac{1}{r'}f_{lJ}(r')\,\mathrm{d}\boldsymbol{r}'\,\mathrm{d}\boldsymbol{\rho}_3\,\mathrm{d}\boldsymbol{\rho}_4$$
$$= \left(\frac{16}{15}\right)^3\left(\frac{3}{2}\right)^3\left(\frac{\alpha}{3\pi}\right)^{3/2}\frac{8M}{5\hbar^2}\sum_{\kappa=1}^{2}\sum_{\nu=1}^{4}2(w-2h)\nu$$
$$\times\frac{^{LL}U_\kappa^\nu}{1+c^2}\,\frac{75\pi}{4\alpha-3^{LL}\mu_\kappa^\nu}\int\exp\left\{-\frac{8}{75}(17\alpha+6^{LL}\mu_\kappa^\nu)(r^2+r'^2)\right\}$$
$$\times\left[\left\{6-2\zeta(r^2+r'^2)-\left(\frac{\omega}{\lambda_{LL}(\kappa)}\right)^2 l(l-1)-4\zeta\frac{l}{\lambda_{LL}(\kappa)}\right\}\mathscr{B}_{l+1/2}\{\lambda_{LL}(\kappa)rr'\}\right.$$
$$\left.+2\left(2\zeta-\frac{\omega^2}{\lambda_{LL}(\kappa)}\right)rr'\mathscr{B}_{l+3/2}\{\lambda_{LL}(\kappa)rr'\}\right]f_{lJ}(r')\,\mathrm{d}r'.$$

Hence

$$^{Q}\mathscr{K}_l(r,r') = \left(\frac{16}{15}\right)^3\left(\frac{3\alpha}{\pi}\right)^{3/2}\frac{M}{\hbar^2}\sum_{\kappa=1}^{2}\sum_{\nu=1}^{4}(2w-4h)\nu\frac{^{LL}U_\kappa^\nu}{1+c^2}$$
$$\times\left(\frac{15\pi}{4\alpha-3^{LL}\mu^\nu}\right)\exp\left\{-\frac{8}{75}(17\alpha+6^{LL}\mu_\kappa^\nu)(r^2+r'^2)\right\}$$
$$\times\left[\left\{6-2\zeta(r^2+r'^2)-\left(\frac{\omega}{\lambda_{LL}(\kappa)}\right)^2 l(l-1)-4\zeta\frac{l}{\lambda_{LL}(\kappa)}\right\}\right.$$
$$\left.\times\mathscr{B}_{l+3/2}\{\lambda_{LL}(\kappa)rr'\}+2\left(2\zeta-\frac{\omega^2}{\lambda_{LL}(\kappa)}\right)rr'\mathscr{B}_{l+3/2}\{\lambda_{LL}(\kappa)rr'\}\right].$$

5.3. *The* S–D *and the* D–S *terms*

The energy terms give the contribution.

5.3.1. *The potential terms*

$$
\begin{aligned}
&{}^{T}\mathscr{K}_{S-D}(r,r')+{}^{T}\mathscr{K}_{D-S}(r,r')\\
&\quad=\frac{8M}{5\hbar^2}2\int \psi_S^*\bar{\chi}_{IJ}^m(1)\sum_{i>j}^{5} S_{ij}{}^{T}U_{ij}\psi_D\chi_{IJ}^m(2)\{f_{IJ}(r')/r'\}\,\mathrm{d}_{\tau-1}r\,\mathrm{d}\Omega\\
&\qquad+\frac{8M}{5\hbar^2}2\int \psi_D^*\bar{\chi}_{IJ}^m(1)\sum_{i>j}^{5} S_{ij}{}^{T}U_{ij}\psi_S\chi_{IJ}^m(2)\left\{\frac{f_{IJ}(r')}{r'}\right\}\mathrm{d}_{\tau-1}r\,\mathrm{d}\Omega
\end{aligned}
$$

where

$$
\begin{aligned}
\psi_s&=\frac{N_S}{(1+c^2)^{1/2}}\exp\left(-\frac{\alpha}{2}\sum_{i>j=2}^{5}r_{ij}^2\right)\\
\psi_D&=\frac{cN_D}{(1+c^2)^{1/2}}\exp\left(-\frac{\beta}{2}\sum_{i>j=2}^{5}r_{ij}^2\right)w_D\\
w_D&=3(\boldsymbol{\sigma}_2\cdot\boldsymbol{r}_{23})(\boldsymbol{\sigma}_4\cdot\boldsymbol{r}_{45})+3(\boldsymbol{\sigma}_2\cdot\boldsymbol{r}_{45})(\boldsymbol{\sigma}_4\cdot\boldsymbol{r}_{23})-2(\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_4)(\boldsymbol{r}_{23}\cdot\boldsymbol{r}_{45})
\end{aligned}
$$

and

$$
N_D^2=\frac{4\beta^2}{45}\left(\frac{16\beta^3}{\pi^3}\right)^{3/2}.
$$

S_{ij} is the usual tensor operator which has already been defined in this paper and U_{ij} is the interaction operator between particles (i,j). Thus

$$
\begin{aligned}
{}^{T}\mathscr{K}_l'(r,r')+{}^{T}\mathscr{K}_l''(r,r')&=-\frac{64}{15}\left(\frac{3}{5\pi}\right)^{1/2}\frac{2^6\alpha^{9/4}\beta^{13/4}b^{1/2}d}{(\alpha+\beta)^6}\frac{c}{1+c^2}\\
&\quad\times\frac{8M}{5\hbar^2}\sum_{i=1}^{2}\sum_{\kappa=1}^{2}\sum_{\nu=3}^{4}(w+m)_\nu{}^{T}U_\kappa^\nu\exp\{-(\gamma^{(i)}r^2+\delta^{(i)}r'^2)\}\\
&\quad\times\left[\left\{S_l^{(i)}-3\left(\frac{16}{15}\right)^2(\alpha+\beta)(T_l^{(i)}r^2+T_l'^{(i)}r'^2)\right.\right.\\
&\quad\left.+9\left(\frac{16}{15}\right)^4(\alpha+\beta)^2(U^{(i)}r^4+2U'^{(i)}r^2r'^2+U''^{(i)}r'^4)\right\}\\
&\quad\times\mathscr{B}_{l+1/2}\{\lambda_T(\kappa)rr'\}+3\left(\frac{16}{15}\right)^2(\alpha+\beta)rr'\\
&\quad\times\left\{-V^{(i)}+3\left(\frac{16}{15}\right)^2(\alpha+\beta)(W^{(i)}r^2+W'^{(i)}r'^2)\right\}\\
&\quad\left.\times\mathscr{B}_{l+3/2}\{\lambda_T(\kappa)rr'\}\right]+\text{a similar term with } r\rightleftharpoons r'\\
&\quad+\frac{64}{15}\left(\frac{3}{5\pi}\right)^{1/2}\frac{2^6\alpha^{9/4}\beta^{13/4}d^{7/2}}{(\alpha+\beta)^6}\frac{c}{1+c^2}\frac{24M}{\hbar^2}\\
&\quad\times\sum_{i=1}^{2}\sum_{\kappa=1}^{2}\sum_{\nu=3}^{4}(w+m)_\nu{}^{T}U_\kappa^\nu\exp\{-(\rho^{(i)}r^2+\xi^{(i)}r'^2)\}
\end{aligned}
$$

$$\times \mathscr{B}_{l+1/2}(\tau_{rr}') + 24\left(\frac{3}{5\pi}\right)^{1/2} \frac{2^6\alpha^{9/4}\beta^{13/4}b^{1/2}d^2}{(\alpha+\beta)^6} \|\sigma_l\| \frac{c}{1+c^2}$$

$$\times \frac{M}{\hbar^2} \sum_{i=1}^{2} \sum_{\kappa=1}^{2} \sum_{\nu=1}^{4} (w+m)_\nu \, {}^T U_\kappa^\nu \exp\{-(\gamma^{(i)} r^2 + \delta^{(i)} r'^2)\}$$

$$\times \left[\left\{X_l - 3\left(\frac{16}{15}\right)^2 (\alpha+\beta)(Y^{(i)}r^2 + Y'^{(i)}r'^2)\right\} \mathscr{B}_{l+1/2}\{\lambda_T(\kappa)rr'\}\right.$$

$$\left. -3\left(\frac{16}{15}\right)^2 (\alpha+\beta)rr'Z\mathscr{B}_{l+3/2}\{\lambda_T(\kappa)rr'\}\right].$$

The definition of all the parameters are given in Appendix 4.

6. Phase-shift analysis

As pointed out previously, coupled states do not arise in n–^{4}He collision. Thus, the asymptotic form of the wave functions is of the following form:

$$f_l(x) = k_l\{\sqrt{x}\cos\delta_l J_{l+1/2}(kx) + \sqrt{x}\sin\delta_l J_{-l-1/2}(kx)\} \tag{6.1}$$

where k_l is the constant amplitude, δ_l is the phase-shift and $J_{l+1/2}$, $J_{-l-1/2}$ are the

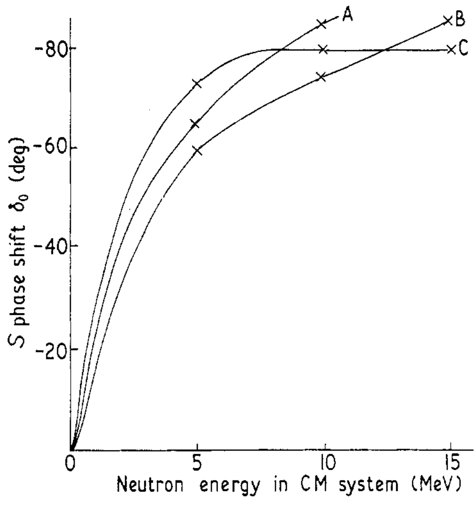

Figure 3. S phase shift in n–^{4}He collisions. A, Seagrave 1953; B, Satchler *et al.* 1968; C, our results.

Bessel functions of half-integral order (angular momentum l), k is the wave number of the incident neutron given by

$$k^2 = \frac{8ME_n}{5\hbar^2} \tag{6.2}$$

where M is the mass of a nucleon and E_n is the energy of the incident particle in CM units.

Given the above formula for $\bar{f}_l(x)$ at two neighbouring points, say r_1 and r_2, we have

$$\frac{\bar{f}_l(r_2)}{\bar{f}_l(r_1)} = \frac{\sqrt{r_2}\{\cos\delta_l J_{l+1/2}(kr_2) + \sin\delta_l J_{-l-1/2}(kr_2)\}}{\sqrt{r_1}\{\cos\delta_l J_{l+1/2}(kr_1) + \sin\delta_l J_{-l-1/2}(kr_1)\}} \tag{6.3}$$

which immediately reduces to

$$\tan\delta_l = -\frac{\bar{f}_l(r_2)\sqrt{r_1}J_{l+1/2}(kr_1) - \bar{f}_l(r_1)\sqrt{r_2}J_{l+1/2}(kr_2)}{\bar{f}_l(r_2)\sqrt{r_1}J_{-l-1/2}(kr_1) - \bar{f}_l(r_1)\sqrt{r_2}J_{l+1/2}(kr_2)}. \tag{6.4}$$

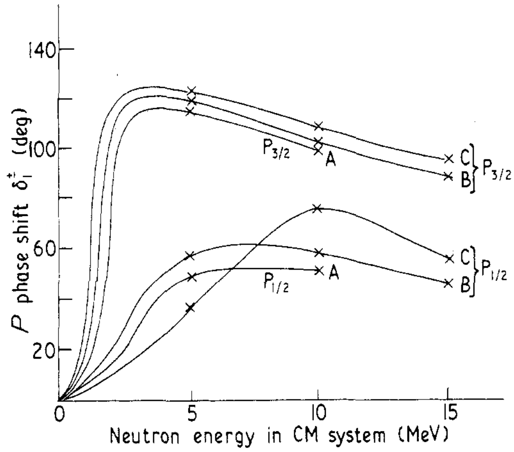

Figure 4. P phase shift in n–^{4}He collisions. A, B and C as for figure 3.

The amplitude k_l corresponding to each phase-shift δ_l can now be found by writing

$$k_l = \bar{f}_l(r_1)/\{\cos\delta_l\sqrt{r_1}J_{l+1/2}(kr_1) + \sin\delta_l\sqrt{r_1}J_{-l-1/2}(kr_1)\}. \tag{6.5}$$

Table 1. The δ_0, δ_1^- and δ_1^+ phases

Author	Energy E_n (MeV)	$k^2 = (8M/5\hbar^2)E_n$	k	δ_0 phases (rad)	δ_1^- phases (rad)	δ_1^+ phases (rad)
Our result	5	0·193	0·440	−1·364	0·628	−1·018
Seagrave 1953	5			−1·117	0·838	−1·082
Satchler *et al.* 1968	5			−1·047	0·995	−1·100
Our result	10	0·386	0·622	−1·368	1·327	−1·369
Seagrave 1953	10			−1·379	0·873	−1·396
Satchler *et al.* 1968	10			−1·358	1·028	−1·344
Our result	15	0·579	0·761	−1·379	1·014	−1·425
Satchler *et al.* 1968	15			−1·606	0·855	−1·552

The phase δ_0 is calculated for the interval of integration $h = 0{\cdot}2$ fm and for $l = 0$ and is given in radians between the range $-\pi/2 \leqslant \delta \leqslant \pi/2$.

The phase δ_1^- is calculated for the interval of integration $h = 0{\cdot}2$ fm and for $l = 1$ and is given in radians between the range $-\pi/2 \leqslant \delta \leqslant \pi/2$.

The phase δ_1^+ is calculated for the interval of integration $h = 0{\cdot}2$ fm and for $l = 1$ and is given in radians between the range $-\pi/2 \leqslant \delta \leqslant \pi/2$.

Table 2

λ	ν	U_1	μ_1	U_2	μ_2
Central	1	−57·301	0·781	−227·752	3·091
	2	−19·519	1·047	−45·081	2·860
	3	−150·433	0·961	−1331·301	3·569
	4	−9·768	0·378	2171·754	4·094
Spin–orbit	1	8·720	0·956	38·279	3·351
	2	−220·659	1·826	−1926·808	4·712
Tensor	1	−72·680	0·848	−2230·868	3·149
	2	14·016	0·737	231·447	2·755
Quadratic spin–orbit	1	36·934	1·620	673·374	4·909
	2	11·670	1·964	−2332·307	9·035
	3	5·050	1·387	−61·792	4·380
	4	−129·299	2·032	−4459·021	5·751

U_K are given in MeV and μ_K in fm^{-2}.

7. Conclusions

The phase-shifts for the elastic scattering of neutrons by alpha particles were calculated for 5, 10 and 15 MeV incident neutron energies (CM system) and for the values of the angular momentum $0 \leqslant l \leqslant 1$.

Our phase-shifts were compared with the experimental phase shifts obtained by Seagrave (1953) and Satchler *et al.* (1968) and found to be in good agreement apart from the δ_1^--phases.

As shown by Omojola (1968—unpublished), the nuclear interaction used gives a binding energy of −26·038 MeV for the alpha particles as compared with the experimental value of −28·2 MeV.

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Appendix 1. Coordinates and transformations

In figure 5 the incident neutron is labelled as the particle 1 and has the position vector $\boldsymbol{r}$. The other neutrons are labelled 2 and 3 and the protons 4 and 5 with respective position vectors $\boldsymbol{r}_2$, $\boldsymbol{r}_3$, $\boldsymbol{r}_4$ and $\boldsymbol{r}_5$

We define $\boldsymbol{r}$ as the coordinate of particle 1 relative to the centre of mass (CM) of the other four and $\boldsymbol{r}'$ as the coordinate of particle 2 relative to the CM of the other four.

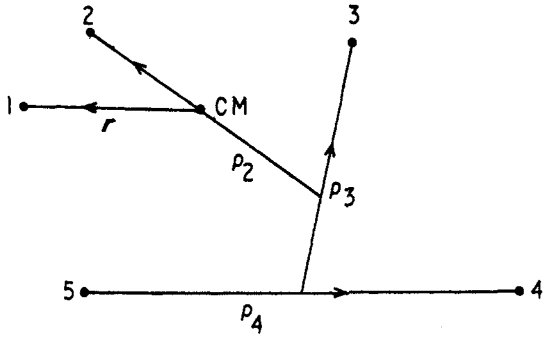

Figure 5.

Choice of coordinates (see figure 5)

$$\begin{aligned} \boldsymbol{r} &= \boldsymbol{r}_1 - \tfrac{1}{4}(\boldsymbol{r}_2+\boldsymbol{r}_3+\boldsymbol{r}_4+\boldsymbol{r}_5) \\ \boldsymbol{r}' &= \boldsymbol{r}_2 - \tfrac{1}{4}(\boldsymbol{r}_1+\boldsymbol{r}_3+\boldsymbol{r}_4+\boldsymbol{r}_5) \\ \boldsymbol{\rho}_3 &= \boldsymbol{r}_3 - \tfrac{1}{2}(\boldsymbol{r}_4+\boldsymbol{r}_5) \\ \boldsymbol{\rho}_4 &= \boldsymbol{r}_4 - \boldsymbol{r}_5. \end{aligned}$$

We define

$$\begin{aligned} \boldsymbol{s} &= \tfrac{4}{15}(4\boldsymbol{r}+\boldsymbol{r}') = \boldsymbol{r}_1 - \tfrac{1}{3}(\boldsymbol{r}_3+\boldsymbol{r}_4+\boldsymbol{r}_5) \equiv \boldsymbol{\rho}_1 \text{ etc.} \\ \boldsymbol{s}' &= \tfrac{4}{15}(\boldsymbol{r}+4\boldsymbol{r}') = \boldsymbol{r}_3 - \tfrac{1}{3}(\boldsymbol{r}_3+\boldsymbol{r}_4+\boldsymbol{r}_5) \equiv \boldsymbol{\rho}_2. \end{aligned}$$

In the calculation for the indirect interaction the transformation used for the volume integration is $d\tau_{-1} = d\boldsymbol{r}_{2345} = (16/15)^3\, d\boldsymbol{r}'\, d\boldsymbol{\rho}_3\, d\boldsymbol{\rho}_4$ with $d\boldsymbol{r}' = r'^2\, dr'\, d\Omega'$, Ω' being the solid angle of $\boldsymbol{r}'$

Appendix 2. Integration formulae and spherical Bessel functions

The direct integrations

In the calculation of the direct integrations, the following formulae are used:

$$K\int \exp\{-\lambda(\boldsymbol{R}-\gamma\boldsymbol{r})^2\}\, d\boldsymbol{R} = 1$$

$$K\int \boldsymbol{R} \exp\{-\lambda(\boldsymbol{R}-\gamma\boldsymbol{r})^2\}\, d\boldsymbol{R} = \gamma\boldsymbol{r}$$

$$K\int R^2 \exp\{-\lambda(\boldsymbol{R}-\gamma\boldsymbol{r})^2\}\, d\boldsymbol{R} = 3/2\lambda+\gamma^2 r^2$$

$$K\int R^3 \exp\{-\lambda(\boldsymbol{R}-\gamma\boldsymbol{r})^2\}\, d\boldsymbol{R} = (5/2\lambda+\gamma^2 r^2)\gamma\boldsymbol{r}$$

$$K\int R^4 \exp\{-\lambda(\boldsymbol{R}-\gamma\boldsymbol{r})^2\}\, d\boldsymbol{R} = 15/4\lambda^2+(5/\lambda)\gamma^2 r^2+\gamma^4 r^4$$

$$K\int R^6 \exp\{-\lambda(\boldsymbol{R}-\gamma\boldsymbol{r})^2\}\,\mathrm{d}\boldsymbol{R} = 105/8\lambda^3+(105/4\lambda^2)\gamma^2r^2+(21/2\lambda)\gamma^4r^4+\gamma^6r^6$$

$$K\int (\boldsymbol{R}\times\boldsymbol{A})\,.\,(\boldsymbol{R}\times\boldsymbol{B})\exp\{-\lambda(\boldsymbol{R}-\gamma\boldsymbol{r})^2\}\,\mathrm{d}\boldsymbol{R}$$
$$= \gamma^2(\boldsymbol{r}\times\boldsymbol{A})\,.\,(\boldsymbol{r}\times\boldsymbol{B})+(1/\lambda)(\boldsymbol{A}\,.\,\boldsymbol{B})$$

$$K\int R^2(\boldsymbol{A}\,.\,\boldsymbol{R})(\boldsymbol{B}\,.\,\boldsymbol{R})\exp\{-\lambda(\boldsymbol{R}-\gamma\boldsymbol{r})^2\}\,\mathrm{d}\boldsymbol{R}$$
$$= (7/2\lambda+\gamma^2r^2)\gamma^2(\boldsymbol{A}\,.\,\boldsymbol{r})(\boldsymbol{B}\,.\,\boldsymbol{r})+(5/2\lambda+\gamma^2r^2)(1/2\lambda)\times(\boldsymbol{A}\,.\,\boldsymbol{B})$$

$$K\int (\boldsymbol{A}\,.\,\boldsymbol{R})(\boldsymbol{B}\,.\,\boldsymbol{R})(\boldsymbol{C}\,.\,\boldsymbol{R})(\boldsymbol{D}\,.\,\boldsymbol{R})\exp\{-\lambda(\boldsymbol{R}-\gamma\boldsymbol{r})^2\}\,\mathrm{d}\boldsymbol{R}$$
$$= \gamma^4(\boldsymbol{A}\,.\,\boldsymbol{r})(\boldsymbol{B}\,.\,\boldsymbol{r})(\boldsymbol{C}\,.\,\boldsymbol{r})(\boldsymbol{D}\,.\,\boldsymbol{r})+(\gamma^2/2\lambda)\times\{(\boldsymbol{A}\,.\,\boldsymbol{B})(\boldsymbol{C}\,.\,\boldsymbol{r})(\boldsymbol{D}\,.\,\boldsymbol{r})+(\boldsymbol{A}\,.\,\boldsymbol{C})(\boldsymbol{B}\,.\,\boldsymbol{r})(\boldsymbol{D}\,.\,\boldsymbol{r})+(\boldsymbol{A}\,.\,\boldsymbol{D})(\boldsymbol{B}\,.\,\boldsymbol{r})(\boldsymbol{C}\,.\,\boldsymbol{r})+(\boldsymbol{B}\,.\,\boldsymbol{C})(\boldsymbol{A}\,.\,\boldsymbol{r})(\boldsymbol{D}\,.\,\boldsymbol{r})+(\boldsymbol{B}\,.\,\boldsymbol{D})(\boldsymbol{A}\,.\,\boldsymbol{r})(\boldsymbol{C}\,.\,\boldsymbol{r})+(\boldsymbol{C}\,.\,\boldsymbol{D})(\boldsymbol{A}\,.\,\boldsymbol{r})(\boldsymbol{B}\,.\,\boldsymbol{r})\}+(1/4\lambda^2)\{(\boldsymbol{A}\,.\,\boldsymbol{B})(\boldsymbol{C}\,.\,\boldsymbol{D})+(\boldsymbol{A}\,.\,\boldsymbol{C})(\boldsymbol{B}\,.\,\boldsymbol{D})+(\boldsymbol{A}\,.\,\boldsymbol{D})(\boldsymbol{B}\,.\,\boldsymbol{C})\}$$

$$K\int R^4(\boldsymbol{A}\,.\,\boldsymbol{R})(\boldsymbol{B}\,.\,\boldsymbol{R})\exp\{-\lambda(\boldsymbol{R}-\gamma\boldsymbol{r})^2\}\,\mathrm{d}\boldsymbol{R}$$
$$= \{\gamma^4r^4+(9/\lambda)\gamma^2r^2+(63/4\lambda^2)\gamma^2(\boldsymbol{A}\,.\,\boldsymbol{r})(\boldsymbol{B}\,.\,\boldsymbol{r})+(1/2\lambda)(\gamma^4r^4+(7/\lambda)\gamma^2r^2+35/4\lambda^2)(\boldsymbol{A}\,.\,\boldsymbol{B})$$

$$K\int (\boldsymbol{A}\,.\,\boldsymbol{R})^3(\boldsymbol{B}\,.\,\boldsymbol{R})^2\exp\{-\lambda(\boldsymbol{R}-\gamma\boldsymbol{r})^2\}\,\mathrm{d}\boldsymbol{R}$$
$$= \gamma^5(\boldsymbol{A}\,.\,\boldsymbol{r})^3(\boldsymbol{B}\,.\,\boldsymbol{r})^2+(\gamma^3/2\lambda)\{3A^2(\boldsymbol{B}\,.\,\boldsymbol{r})^2+B^2(\boldsymbol{A}\,.\,\boldsymbol{r})^2+6(\boldsymbol{A}\,.\,\boldsymbol{B})(\boldsymbol{A}\,.\,\boldsymbol{r})(\boldsymbol{B}\,.\,\boldsymbol{r})\}(\boldsymbol{A}\,.\,\boldsymbol{r})+(3\gamma/4\lambda^2)\{A^2B^2\times(\boldsymbol{A}\,.\,\boldsymbol{r})+2(\boldsymbol{A}\,.\,\boldsymbol{B})^2(\boldsymbol{A}\,.\,\boldsymbol{r})+2A^2(\boldsymbol{A}\,.\,\boldsymbol{B})(\boldsymbol{B}\,.\,\boldsymbol{r})\}$$

$$K\int S_{12}(R^2)\exp\{-\lambda(\boldsymbol{R}-\gamma\boldsymbol{r})^2\}\,\mathrm{d}\boldsymbol{R}$$
$$= \gamma^2S_{12}(r^2)$$

$$K\int (\boldsymbol{A}\,.\,\boldsymbol{R})S_{12}(R^2)\exp\{-\lambda(\boldsymbol{R}-\gamma\boldsymbol{r})^2\}\,\mathrm{d}\boldsymbol{R}$$
$$= \gamma^2(\boldsymbol{A}\,.\,\boldsymbol{r})S_{12}(r^2)+(\gamma/\lambda)S_{12}(\boldsymbol{A}\,.\,\boldsymbol{r})$$

$$K\int R^2S_{12}(R^2)\exp\{-\lambda(\boldsymbol{R}-\gamma\boldsymbol{r})^2\}\,\mathrm{d}\boldsymbol{R}$$
$$= \{(7/2\lambda)+\gamma^2r^2)\}\gamma^2S_{12}(r^2)\ \text{etc.}$$

where

$$K = (\lambda/\pi)^{3/2}$$

and

$$S_{12}(\boldsymbol{A}\,.\,\boldsymbol{B}) = S_{12}(\boldsymbol{B}\,.\,\boldsymbol{A}) = 3(\boldsymbol{\sigma}_1\,.\,\boldsymbol{A})(\boldsymbol{\sigma}_2\,.\,\boldsymbol{B})-(\boldsymbol{\sigma}_1\,.\,\boldsymbol{\sigma}_2)(\boldsymbol{A}\,.\,\boldsymbol{B}).$$

The indirect integrations

The integrations over spherical harmonics in the indirect interaction calculation depend on the use of the following well-known formula:

$$\int \mathscr{Y}_{lm}(\theta,\phi)\exp(-\delta\boldsymbol{r}\,.\,\boldsymbol{r}')\,\mathrm{d}\Omega = \frac{4\pi}{\delta rr'}\mathscr{Y}_{lm}(\theta',\phi')\mathscr{B}_{l+1/2}(\delta rr') \qquad \text{(A2.1)}$$

where

$$\left.\begin{aligned} \boldsymbol{r} &= (r, \Omega) = (r, \theta, \phi) \\ \boldsymbol{r}' &= (r', \Omega') = (r', \theta', \phi') \\ \mathrm{d}\Omega &= \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \end{aligned}\right\} \tag{A2.2}$$

and

$$\mathscr{B}_{l+1/2}(x) = i^{l-1/2}(\pi x/2)^{1/2} \mathrm{J}_{l+1/2}(ix).$$

This formula is well known and has been used by many workers in several related scattering calculations (e.g. Sugie *et al.* 1957 and Hochberg *et al.* 1968—unpublished). The derivation of the above formula (A2.1) can be found in several texts (e.g. Whittaker and Watson 1952). The first few values of the above Bessel function are given below:

$$\left.\begin{aligned} \mathscr{B}_{1/2}(x) &= \sinh x \\ \mathscr{B}_{3/2}(x) &= \frac{\sinh x}{x} - \cosh x. \end{aligned}\right\} \tag{A2.3}$$

The recurrence relations between the functions are (from e.g. Whittaker and Watson 1952)

$$\frac{2l+1}{x} \mathscr{B}_{l+1/2}(x) = \mathscr{B}_{l+3/2}(x) - \mathscr{B}_{l-1/2}(x) \tag{A2.4}$$

and

$$\begin{aligned} \mathscr{B}'_{l+1/2}(x) &= \frac{\mathrm{d}}{\mathrm{d}x} \mathscr{B}_{l+1/2}(x) \\ &= \frac{l+1}{x} \mathscr{B}_{l+1/2}(x) - \mathscr{B}_{l+3/2}(x). \end{aligned} \tag{A2.5}$$

From equations (A2.4) and (A2.5) we deduce the following result:

$$\mathscr{B}''_{l+1/2}(x) = \left\{\frac{l(l+1)}{x^2} + 1\right\} \mathscr{B}_{l+1/2}(x). \tag{A2.6}$$

Also, from equation (A2.3) we deduce the following relation:

$$\mathscr{B}_{l+1/2}(-x) = (-1)^{l+1} \mathscr{B}_{l+1/2}(x). \tag{A2.7}$$

Appendix 3. Numerical constants

The alpha radial wave functions

The S- and D-state alpha radial wave function parameters α, β and c are determined from the variational calculation of the binding energy of ^{4}He by the direct search method (Omojola 1968—unpublished).

The value of α obtained from the variational calculation of the binding energy of ^{4}He is not used in our calculations in this work, instead α is taken to be $0{\cdot}140\ \mathrm{fm}^{-2}$. This value assigned to α is established from high-energy e–^{4}He scattering. The numerical values of these parameters are listed below:

$$\alpha = 0{\cdot}140$$

$$\beta = 0{\cdot}262$$

and

$$c = 0{\cdot}145.$$

Both α and β have the same unit of fm^{-2} (i.e. 10^{26} cm^{-2}). Other constants used are given below:

$$M/\hbar^2 = 0{\cdot}024148 \times 10^{26}\ \mathrm{cm}^{-2}\ \mathrm{MeV}^{-1}$$

$$E_\alpha\ (\text{experimental}) = -28{\cdot}2\ \mathrm{MeV}$$

$$E_\alpha\ (\text{variational}) = -26{\cdot}038\ \mathrm{MeV}$$

and

$$e^2 = 1{\cdot}445 \times 10^{-13}\ \mathrm{cm\ MeV}.$$

The nuclear interaction

The interactions defined in various sections are defined in general by

$$V(r) \sim \sum_{\kappa=1}^{2} {}^{\lambda}U_\kappa^{(\nu)} \exp\{-{}^{\lambda}\mu_\kappa^{(\nu)} r^2\}$$

and in the case of the tensor force only by

$$V(r) \sim r^2 \sum_{\kappa=1}^{2} {}^{\lambda}U_\kappa^{(\nu)} \exp\{-{}^{\lambda}\mu_\kappa^{(\nu)} r^2\}.$$

The numerical values of U_κ and μ_κ are given in table 2.

Appendix 4. Collection of parameters

The parameters defined in the direct interaction calculation are as follows:

S–S

$$P_c(\kappa) \equiv \frac{16\alpha}{16\alpha + 3\,{}^{c}\mu_\kappa^{(\nu)}}$$

$$P_{Ls}(\kappa) \equiv \frac{16\alpha}{16\alpha + 3\,{}^{Ls}\mu_\kappa^{(\nu)}}$$

$$\gamma_{LL}(\kappa) \equiv \frac{4\,{}^{LL}\mu_\kappa^{(\nu)}}{16\alpha + 3\,{}^{LL}\mu_\kappa^{(\nu)}}$$

$$\sigma \equiv 3\alpha\left[\frac{3\alpha}{\lambda'_{LL}(\kappa)} - \left\{1 + \frac{5}{4}\gamma_{LL}(\kappa)\right\}\left\{1 - \frac{3}{4}\gamma_{LL}(\kappa)\right\}\right]$$

$$\eta \equiv \frac{15}{16}\left[\left\{1 + \frac{5}{4}\gamma_{LL}(\kappa)\right\}\left\{1 - \frac{3}{4}\gamma_{LL}(\kappa)\right\} - \frac{3\alpha}{\lambda'_{LL}(\kappa)}\right]$$

$$\lambda'_{LL}(\kappa) \equiv \frac{3}{16}(16\alpha + 3\,{}^{LL}\mu_\kappa^{(\nu)}).$$

In the indirect calculation the following parameters are defined.

S–S

$$\lambda_c(\kappa) = \tfrac{32}{75}(4\alpha - 3\,{}^{c}\mu_\kappa^{(\nu)})$$

$$\lambda_{Ls}(\kappa) = \tfrac{32}{75}(4\alpha - 3\,{}^{Ls}\mu_\kappa^{(\nu)})$$

$$\lambda_{LL}(\kappa) \equiv \tfrac{32}{75}(4\alpha - 3\,{}^{LL}\mu_\kappa^{(\nu)})$$

$$\eta_c \equiv \frac{6\alpha}{6\alpha + {}^c\mu_\kappa^{(\nu)}}$$

$$\epsilon \equiv \frac{128\alpha}{75}$$

$$\zeta_c(\kappa) \equiv \frac{128\alpha}{25}\left(\frac{2\alpha + {}^c\mu_\kappa^{(\nu)}}{6\alpha + {}^c\mu_\kappa^{(\nu)}}\right)$$

$$\omega \equiv \frac{128\alpha}{15}$$

$$\zeta_{LL}(\kappa) \equiv \tfrac{32}{75}\,(16\alpha + 3\,{}^{LL}\mu_\kappa^{(\nu)})$$

S–D

$$a \equiv \frac{{}^T\mu_\kappa^{(\nu)}}{3(\alpha+\beta) + {}^T\mu_\kappa^{(\nu)}}$$

$$b \equiv 3(\alpha+\beta)/\{3(\alpha+\beta) + {}^T\mu_\kappa^{(\nu)}\}$$

$$d \equiv (\alpha+\beta)/(\alpha+\beta+{}^T\mu_\kappa^{(\nu)})$$

$$\lambda_T \equiv 3(\tfrac{8}{15})^2\,(1+2a)(\alpha+\beta)$$

$$\gamma^{(1)} \equiv \tfrac{24}{225}\,(\alpha+16\beta) + 3(\tfrac{16}{15})^2\,(\alpha+\beta)a$$

$$\delta^{(1)} \equiv \tfrac{24}{225}\,(16\alpha+\beta) + \tfrac{3}{16}\,(\tfrac{16}{15})^2\,(\alpha+\beta)a$$

$$S_l^{(1)} \equiv \tfrac{5}{2}b^2 + \tfrac{20}{3}\,lbd(2a+b) + 8l(l-1)ad^2(a+\tfrac{2}{3}b)$$

$$T_l^{(1)} \equiv \tfrac{10}{3}\,b^2(2a+b) + 8labd(a+\tfrac{2}{3}b)$$

$$T_l'^{(1)} \equiv \tfrac{5}{24}\,b^2(2a+b) + \tfrac{1}{2}labd(a+\tfrac{2}{3}b)$$

$$U^{(1)} \equiv 2ab^2(a+\tfrac{2}{3}b)$$

$$U'^{(1)} = \tfrac{3}{16}\,U^{(1)}$$

$$U''^{(1)} = \tfrac{1}{256}\,U^{(1)}$$

$$V^{(1)} \equiv \tfrac{5}{3}b^2(2a+b) - 4abd(a+\tfrac{2}{3}b)$$

$$W^{(1)} = U^{(1)}$$

$$W'^{(1)} = \tfrac{1}{16}\,U^{(1)}$$

$$\rho^{(1)} \equiv \tfrac{24}{225}\,(\alpha+16\beta)$$

$$\xi^{(1)} \equiv \tfrac{24}{225}\,(16\alpha+\beta)$$

$$\tau \equiv (\tfrac{3}{4})(\tfrac{16}{15})^2\,(\alpha+\beta)$$

$$Y'^{(1)} \equiv \tfrac{16}{15}\,ab$$

$$Y^{(1)} = \tfrac{1}{16}\,Y'^{(1)}$$

D–S

$$\gamma^{(2)} \equiv \tfrac{24}{225}\,(16\alpha+\beta) + 3(\tfrac{16}{15})^2\,(\alpha+\beta)a$$

$$\delta^{(2)} \equiv \tfrac{24}{225}\,(\alpha+16\beta) + \tfrac{3}{16}\,(\tfrac{16}{15})^2\,(\alpha+\beta)a$$

$$S_l^{(2)} \equiv \tfrac{5}{6}b^2 + \tfrac{5}{6}lbd(25a+17b) + 2l(l-1)ad^2(7a+\tfrac{17}{3}\,b)$$

$$T_l^{(2)} \equiv \tfrac{5}{6}b^2(5a+b) + labd(9a+\tfrac{19}{3}\,b)$$

$$T_l'^{(2)} \equiv \tfrac{5}{24} b^2(5a+4b)+\tfrac{1}{16} labd(19a+\tfrac{49}{3} b)$$

$$U^{(2)} \equiv ab^2(a+\tfrac{1}{3}b)$$

$$U'^{(2)} \equiv \tfrac{1}{32} ab^2(21a+17b)$$

$$U''^{(2)} \equiv \tfrac{1}{128} ab^2\left(3a+\frac{8b}{3}\right)$$

$$V^{(2)} \equiv \tfrac{5}{24} b^2(25a+17b)-abd(7a+\tfrac{17}{3} b)$$

$$W^{(2)} \equiv \tfrac{1}{4}ab^2(9a+\tfrac{19}{3} b)$$

$$W'^{(2)} \equiv \tfrac{1}{64} ab^2(19a+\tfrac{49}{3} b)$$

$$X_l \equiv \tfrac{8}{3} b+\tfrac{32}{15} lad$$

$$Y^{(2)} \equiv \tfrac{16}{15} ab$$

$$Y'^{(2)} = \tfrac{1}{16} Y^{(2)}$$

$$Z = \tfrac{1}{2} Y^{(2)}$$

$$\rho^{(2)} \equiv \tfrac{24}{225} (16\alpha+\beta)$$

$$\xi^{(2)} \equiv \tfrac{24}{225} (\alpha+16\beta)$$

$$\|\boldsymbol{\sigma}_l\| = \begin{cases} 0 & \text{for} \quad J=\tfrac{1}{2}, \quad l=0 \\ l & \text{for} \quad J=l+\tfrac{1}{2} \\ -(l+1) & \text{for} \quad J=l-\tfrac{1}{2}. \end{cases}$$

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